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A Note on Bernstein's Theorems

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There is given a completion to Theorem 3.3 of [11] by showing that on compact subsets of $\mathbb{R}^N$ (or $\mathbb{C}^N$) preserving Markov's inequality, some speed of polynomial approximation leads to Lipschitz- and Zygmund-type classes of functions. © 1995 Academic Press, Inc.

The following basic notation will be used throughout the paper. $\mathbb{K}$ is a field of either real: $\mathbb{R}$ or complex: $\mathbb{C}$ scalars, E is a non-empty compact subset of $\mathbb{K}^N$, $N \geqslant 1$. The supremum norm of a $\mathbb{K}$-valued function f over $A \subset \mathbb{K}^N$ is denoted by $\|f\|_A$, and $\mathscr{P}_n(\mathbb{K}^N)$ is the set of algebraic polynomials in N variables from $\mathbb{K}$ of total degree at most n. Given a function f on A define

$$\operatorname{dist}_A(f, \mathscr{P}_n(\mathbb{K}^N)) := \inf\{\|f-P\|_A : P \in \mathscr{P}_n(\mathbb{K}^N)\}.$$

Then

$$P_{n,A}(f) := \{P \in \mathscr{P}_n(\mathbb{K}^N) : \|f-P\|_A = \operatorname{dist}_A(f, \mathscr{P}_n(\mathbb{K}^N))\}$$

is a nonempty set of best approximations to f in $\mathscr{P}_n(\mathbb{K}^N)$. Also define the geometric distance between $x \in \mathbb{K}^N$ and the set A as

$$\operatorname{dist}(x, A) := \inf\{|x-y| : y \in A\}$$

($|\cdot|$ is the usual Euclidean distance in $\mathbb{K}^N$).

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The main objects of our considerations are the following Lipschitz-type spaces. Given a nonnegative integer k and $\rho \in (0, 1]$, a function $f\colon \mathbb{R}^N \to \mathbb{K}$ is said to belong to the space $\mathrm{Lip}(k+\rho, \mathbb{R}^N)$ iff $f \in C^k(\mathbb{R}^N)$ and there exists a constant $M > 0$ such that for every multi-index $\alpha \in \mathbb{Z}^N_+$,

$$\|D^\alpha f\|_{\mathbb{R}^N} \leqslant M, \quad |\alpha| \leqslant k \tag{1}$$

and

$$\|\Delta_h D^\alpha f\|_{\mathbb{R}^N} \leqslant M|h|^\rho, \quad |\alpha| = k, \ h \in \mathbb{R}^N, \tag{2}$$

where $\Delta_h g(x) := g(x+h) - g(x)$, $x, h \in \mathbb{R}^N$, is the first difference of a function of a function g defined on $\mathbb{R}^N$. Analogously, $f \in \Lambda_{k+\rho}(\mathbb{R}^N)$ iff $f \in C^k(\mathbb{R}^N)$, (1) holds and in (2) Δ_h is replaced by the second difference $\Delta_h^2 := \Delta_h \Delta_h$, i.e.

$$\|\Delta_h^2 D^\alpha f\|_{\mathbb{R}^N} \leqslant M|h|^\rho, \quad |\alpha| = k, \ h \in \mathbb{R}^N, \tag{2'}$$

is fulfilled. The norms in $\mathrm{Lip}(k+\rho, \mathbb{R}^N)$ and $\Lambda_{k+\rho}(\mathbb{R}^N)$ may be defined by taking infima of the constants M in the definitions above. Then $\mathrm{Lip}(k+\rho, \mathbb{R}^N) = \Lambda_{k+\rho}(\mathbb{R}^N)$ with equivalent norms when $0 < \rho < 1$ and $\mathrm{Lip}(k+1, \mathbb{R}^N)$ is strictly contained in $\Lambda_{k+1}(\mathbb{R}^N)$. We refer to [3] for a more detailed discussion on these function spaces and their restriction to general closed sets.

In [11, Theorem 3.3] it was presented a Bernstein-type sufficient condition for a continuous function to have smoothness properties. There was shown in particular that if a compact subset E of $\mathbb{R}^N$ has the property (4) (cf. Lemma 2 below) then $\mathrm{dist}_E(f, \mathscr{P}_n(\mathbb{R}^N)) = O(n^{-r(k+\rho)})$, $k \in \mathbb{Z}_+$, $\rho \in (0, 1)$, implies that f can be extended to a function in $\mathrm{Lip}(k+\rho, \mathbb{R}^N)$. The purpose of this paper is to fill a certain gap that appeared in [11], i.e. we are going to show that if $\rho \in (0, 1]$ then one can construct an extension of f to a function in $\Lambda_{k+\rho}(\mathbb{R}^N)$.

We shall need this known result.

Lemma 1 (see e.g. [8, Lemme IV 3.3]). *There exist positive constants C_α (depending only on $\alpha \in \mathbb{Z}^N_+$) such that for any closed subset E of $\mathbb{R}^N$ and any $\varepsilon > 0$ there exists a function $u_\varepsilon \in C^\infty(\mathbb{R}^N)$ satisfying*

$u_\varepsilon = 1$ in a neighbourhood of E,

$u_\varepsilon = 0$ if $\mathrm{dist}(x, E) \geqslant \varepsilon$,

$0 \leqslant u_\varepsilon \leqslant 1$,

and for every $\alpha \in \mathbb{Z}^N_+$, $|D^\alpha u_\varepsilon(x)| \leqslant C_\alpha \varepsilon^{-|\alpha|}$, $x \in \mathbb{R}^N$.

DEFINITION. A compact non-empty subset E of $\mathbb{K}^N$ is said to preserve the ($\mathbb{K}^N$) global Markov inequality if there exist constants $M>0$ and $r>0$ depending only on E and N, such that

$$\|D^\alpha P\|_E \leqslant M n^{r|\alpha|} \|P\|_E, \tag{3}$$

for all $\alpha \in \mathbb{Z}_+^N$, $n \geqslant 1$ and $P \in \mathscr{P}_n(\mathbb{K}^N)$.

Very general examples of compacta preserving the global Markov inequality (uniformly polynomially cuspidal sets) can be found in [4].

Taylor's formula and the global Markov inequality imply

LEMMA 2 [5, p. 112]. *Let E be a compact subset of $\mathbb{K}^N$ preserving the global Markov inequality. For every $n=1, 2, \ldots$ and every $P \in \mathscr{P}_n(\mathbb{K}^N)$, if $\operatorname{dist}(x, E) \leqslant 1/n^r$, $x \in \mathbb{K}^N$, then*

$$|P(x)| \leqslant M e^N \|P\|_E, \tag{4}$$

with the same M and r as in (3).

With these preliminaries we can state the basic observation.

THEOREM 3. *Suppose $E \subset \mathbb{R}^N$ preserves the global Markov inequality and f is a real-valued function defined on E such that*

$$\operatorname{dist}_E(f, \mathscr{P}_n(\mathbb{R}^N)) = O(1/n^{r(k+\rho)}), \tag{5}$$

where r is given by (3), *k is a non-negative integer and $\rho \in (0, 1]$. Then there exists a function $\tilde{f} \in \Lambda_{k+\rho}(\mathbb{R}^N)$ such that $\tilde{f}|_E = f$.*

Proof. Set $Q_0 = P_1$, $Q_n = P_{2^n} - P_{2^{n-1}}$ where $P_n \in P_{n,E}(f)$. For each n, let $u_n = u_{\varepsilon_n}$ be a $C^\infty(\mathbb{R}^N)$ function obtained from Lemma 1 with $\varepsilon_n = 1/2^{nr}$. Then

$$\tilde{f} := \sum_{n=0}^{\infty} u_n Q_n$$

is an appropriate extension of f to $\mathbb{R}^N$. Since $u_n|_E = 1$ and P_n tends to f uniformly on E we get $\tilde{f}|_E = f$.

Take $\alpha \in \mathbb{Z}_+^N$ and let $E_n := \{x \in \mathbb{R}^N : \operatorname{dist}(x, E) \leqslant \varepsilon_n\}$. Then

$$\begin{aligned} \|D^\alpha(u_n Q_n)\|_{\mathbb{R}^N} &\underset{\textit{Lemma } 1}{=} \|D^\alpha(u_n Q_n)\|_{E_n} \\ &\leqslant \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \|D^{\alpha-\beta} u_n\|_{E_n} \|D^\beta Q_n\|_{E_n} \end{aligned}$$

$$\underset{\text{Lemma 1}}{\leqslant} \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} C_{\alpha-\beta} 2^{nr|\alpha-\beta|} \|D^\beta Q_n\|_{E_n}$$

$$\underset{(4)}{\leqslant} M_1 \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} C_{\alpha-\beta} 2^{nr|\alpha-\beta|} \|D^\beta Q_n\|_E$$

$$\underset{(3)}{\leqslant} M_2 2^{nr|\alpha|} \|Q_n\|_E. \tag{6}$$

By the assumption (5),

$$\|Q_n\|_E \leqslant \|f - P_{2^n}\|_E + \|f - P_{2^{n-1}}\|_E \leqslant \frac{M_3}{2^{nr(k+\rho)}}$$

which together with (6) gives

$$\|D^\alpha(u_n Q_n)\|_{\mathbb{R}^N} \leqslant \frac{M_4}{2^{nr(k-|\alpha|+\rho)}}. \tag{7}$$

Therefore

$$\|D^\alpha \tilde{f}\|_{\mathbb{R}^N} \leqslant \sum_{n=0}^{\infty} \|D^\alpha(u_n Q_n)\|_{\mathbb{R}^N} \leqslant M_5, \quad |\alpha| \leqslant k.$$

That is, $\tilde{f} \in C^k(\mathbb{R}^N)$ and the derivatives of $\tilde{f}$ of order not exceeding k are uniformly bounded which is a necessary condition for $\tilde{f}$ to be in $\Lambda_{k+\rho}(\mathbb{R}^N)$. Finally, to prove (2′) for $\tilde{f}$ we have to estimate the second difference of the derivatives of order k.

Let $\alpha \in \mathbb{Z}_+^N$ such that $|\alpha| = k$ and let $x, h \in \mathbb{R}^N$. Suppose $|h| < 1$ and choose $m \geqslant 1$ such that

$$2^{m-1} < |h|^{-1/r} \leqslant 2^m.$$

We shall estimate

$$\|\Delta_h^2 D^\alpha \tilde{f}\|_{\mathbb{R}^N} \leqslant \left(\sum_{n=0}^{m-1} + \sum_{n=m}^{\infty} \right) \|\Delta_h^2 D^\alpha(u_n Q_n)\|_{\mathbb{R}^N}$$

where

$$\sum_{n=m}^{\infty} \|\Delta_h^2 D^\alpha(u_n Q_n)\|_{\mathbb{R}^N} \underset{(7)}{\leqslant} 4M_4 \sum_{n=m}^{\infty} 2^{-nr\rho} \leqslant M_5 2^{-mr\rho} \leqslant M_5 |h|^\rho.$$

The mean-value theorem used twice and (7) yields

$$\|\Delta_h^2 D^\alpha(u_n Q_n)\|_{\mathbb{R}^N} \leqslant M_6 |h|^2 2^{nr(2-\rho)}.$$

This implies

$$\sum_{n=0}^{m-1} \|\Delta_h^2 D^\alpha(u_n Q_n)\|_{\mathbb{R}^N} \leqslant M_7 |h|^\rho.$$

Hence $|\Delta_h^2 D^\alpha \tilde{f}(x)|/|h|^\rho$ is uniformly bounded on $\mathbb{R}^N$ which is what we need. If $|h| \geqslant 1$, then the estimation above is trivial, and the proof is completed.

In the case of $\mathbb{K} = \mathbb{C}$, Theorem 3 may be reformulated as follows:

THEOREM 4. *Let $E \subset \mathbb{C}^N$ preserve the global Markov inequality. If for a complex function f defined on E it holds*

$$\operatorname{dist}_E(f, \mathscr{P}_n(\mathbb{C}^N)) = O(1/n^{r(k+\rho)}), \tag{5'}$$

where r is given by (3), k is a non-negative integer and $\rho \in (0, 1]$, then there exists a function $\tilde{f} \in \Lambda_{k+\rho}(\mathbb{R}^{2N})$ such that $\tilde{f}|_E = f$.

The proof proceeds along the same lines as the proof of Theorem 3.

A typical approach to Bernstein-Zygmund inverse theorems of approximation theory leads to results that connect the speed of approximation to the smoothness of the approximated function measured by some modulus. The litterature contains numerous definitions of moduli, e.g. the ones for convex subsets A of $\mathbb{K}^N$ defined with help of the k : th difference:

For a $\mathbb{K}$-valued function f on A and a positive integer k the quantity

$$\omega_k(f, \delta) := \sup\{|\Delta_h^k f(x)| : |h| \leqslant \delta, x \in A, \operatorname{dist}(x, \mathbb{K}^N \backslash A) < k|h|\}, \quad \delta > 0,$$

where Δ_h^k is the k-th difference, is called the modulus of continuity of f of order k.

It is not possible to restate directly the above definition of moduli of order greater than one for sets with complicated structure. To avoid problems cuased by the shape of the function domain we propose the following formula.

First, denote by $L_j(\mathbb{K}^N, \mathbb{K})$ the space of j-linear functionals from $\mathbb{K}^N$ to $\mathbb{K}$. If $L \in L_j(\mathbb{K}^N, \mathbb{K})$ and $w \in \mathbb{K}^N$ then, as usual, $Lw^j = L(w, ..., w)$, with the argument w j-times repeated.

Take A to be any nonempty closed subset of $\mathbb{K}^N$ and $f: A \to \mathbb{K}$. For an integer $k \geqslant 2$, the quantity

$$\tilde{\omega}_k(f, \delta) := \sup_{y \in A} \inf_{\substack{L_j \in L_j(\mathbb{K}^N, \mathbb{K}) \\ j=1, ..., k-1}} \sup_{\substack{x \in A \\ |x-y| \leqslant \delta}} \left| f(x) - f(y) - \sum_{j=1}^{k-1} L_j(x-y)^j \right|$$

is said to be the modulus of continuity of f of order k.

Remark. For every integer $k \geqslant 1$ there exist positive constants c_k and C_k such that for every interval $[a, b]$ and for every function f continuous on $[a, b]$ the inequality

$$c_k \omega_k \left(f, \frac{1}{k}(b-a)\right) \leqslant \operatorname{dist}_{[a,b]}(f, \mathscr{P}_{k-1}(\mathbb{R})) \leqslant C_k \omega_k \left(f, \frac{1}{k}(b-a)\right)$$

holds. Therefore $\tilde{\omega}_k$ is a very natural extension of the modulus of continuity ω_k. The left inequality was shown in [6], the right one in [10].

We do not claim originality of the definition of $\tilde{\omega}_k(f, \delta)$. Methods of measuring smoothness of functions given on general subset of $\mathbb{R}^N$ by means of local polynomial approximation was presented in a series of papers by Brudnyi (see e.g. [2] and references there). In the complex plane, Vorob'ev and Polyakov [9] introduced an extension moduli of higher order by means of local interpolation on smooth arcs. Seven years later Tamrazov presented a series of definition [7, pp. 47–49] of moduli of continuity and smoothness for functions defined on general plane sets and discussed thoroughly their approximation properties. The inverse approximation theorem on plane continua that was proved by Lebedev and Tamrazov did not cover the case of integer order of smoothness. To complete this result, Bijvoets, Hoggeven and Korevaar [1] introduced a generalized Zygmund modulus of smoothness, i.e. the modulus of continuity of the second order (cf. [12]), to the case of plane continua with the help of local approximation by polynomials of first degree. We would like to show that $\tilde{\omega}_k(f, \delta)$ yields a very convenient method to reformulate Bernstein- or Zygmund-type inverse theorems in the classical manner even for sets can possess such complicated structure as uniformly polynomially cuspidal sets.

PROPOSITION 5. *Let E be a compact subset of $\mathbb{K}^N$ preserving the global Markov inequality. Suppose that for $f: E \to \mathbb{K}$, (5) if $\mathbb{K} = \mathbb{R}$ or (5′) if $\mathbb{K} = \mathbb{C}$, holds. Then $\tilde{\omega}_{[k+\rho]+1}(f, \delta) = O(\delta^{k+\rho})$.*

Proof. Let P_n and Q_n have similar meaning as in the proof of Theorem 3. For a fixed $y \in E$ and $\delta > 0$ put

$$L_j := \frac{1}{j!} P_{2^m-1}^{(j)}(y) = \frac{1}{j!} \sum_{n=0}^{m-1} Q_n^{(j)}(y),$$

where m satisfies $2^{(m-1)r} < \delta^{-1} \leqslant 2^{mr}$. Then, for $x \in E$ with $|x-y| \leqslant \delta$ we obtain

$$\left| f(x)-f(y)-\sum_{j=1}^{[k+\rho]} L_j(x-y)^j \right|$$
$$\leqslant \sum_{n=m}^{\infty} |Q_n(x)-Q_n(y)| + \sum_{n=0}^{m-1} \left| Q_n(x)-Q_n(y) - \sum_{j=1}^{[k+\rho]} \frac{1}{j!} Q_n^{(j)}(y)(x-y)^j \right|$$
$$\underset{(3)}{\leqslant} 2 \sum_{n=m}^{\infty} \|Q_n\|_E + M \sum_{n:\, 2^n > [k+\rho]}^{m-1} \sum_{[k+\rho]+1 \leqslant |\alpha| \leqslant 2^n} \frac{(\delta 2^{nr})^{|\alpha|}}{\alpha!} \|Q_n\|_E$$
$$\underset{(5),(5')}{\leqslant} C_1 \delta^{k+\rho} + C_2 \delta^{[k+\rho]+1} \sum_{n=0}^{m-1} 2^{nr([\rho]+1-\rho)}$$
$$\leqslant C_3 \delta^{k+\rho}.$$

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